

# Conserved Quantities from the Equations of Motion

(with applications to natural and gauge natural theories of gravitation)

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## Abstract

We present an alternative field theoretical approach to the definition of conserved quantities, based directly on the field equations content of a Lagrangian theory (in the standard framework of the Calculus of Variations in jet bundles). The contraction of the Euler–Lagrange equations with Lie derivatives of the dynamical fields allows one to derive a *variational Lagrangian* for any given set of Lagrangian equations. A two steps algorithmical procedure can be thence applied to the variational Lagrangian in order to produce a general expression for the variation of all quantities which are (covariantly) conserved along the given dynamics. As a concrete example we test this new formalism on Einstein’s equations: well known and widely accepted formulae for the variation of the Hamiltonian and the variation of Energy for General Relativity are recovered. We also consider the Einstein–Cartan (Sciama–Kibble) theory in tetrad formalism and as a by-product we gain some new insight on the Kosmann lift in gauge natural theories, which arises when trying to restore naturality in a gauge natural variational Lagrangian.

## 1 Introduction

A number of physically reasonable geometric definitions of conserved quantities in field theories may be found in literature; in the recent past the issue to define conserved quantities for Lagrangian field theories has been in fact investigated by many authors on the basis of different formalisms. Just to mention a few of them we recall the Lagrangian method based on Noether’s theorem [35, 48, 49, 61, 65], the Hamiltonian approach and the symplectic methods [4, 6, 11, 14, 17, 19, 44, 53, 63], the Hamilton–Jacobi–based techniques [10, 13, 15, 16], and the formulations which are directly based on field equations [5, 50, 64]. These are

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just some of the many references which are most relevant for our purposes, whereby other important literature is quoted. Since a satisfactory review is out of the scope of this paper we apologize for not being able to quote all authors and sources of information and we refer the reader to the quoted papers and references therein.

In view of the fact that there exists in literature a widespread family of somewhat unrelated definitions it would be thence rather important to frame, as much as possible, all possible definitions under a unique and single method of construction. Very far from reaching such a fundamental goal, the scope of the present paper falls nevertheless into this line of thought. Our hope is to shed at least some new light on a fully covariant description of conserved quantities in gauge natural field theories. We choose the so-called *gauge natural* framework [21, 56] since it represents a rich geometrical structure which encompasses all Lagrangian field theories relevant to fundamental physics (see also [25, 27, 41, 59]); this modern formalism is in fact well suited to describe in a single mathematical context both natural theories (the most famous of which is General Relativity) as well as gauge theories together with their possible couplings with Bosonic and Fermionic matter. This setting becomes therefore a powerful tool to frame conserved quantities in a geometric setting which is general enough to include all relevant physical fields: gauge natural theories play in fact a privileged role to this purpose since they are defined, from the very beginning, on the basis of the covariance properties of the fields under the appropriate transformation groups.

We point out that the gauge natural formalism is essentially a Lagrangian approach. Nevertheless field equations are more fundamental than the Lagrangian itself in the description of dynamics. Field equations, in fact, rule out those field configurations which are physically admissible and dictate their dynamical evolution; accordingly we would prefer to focus our attention directly on the equations of motion.

However, it has to be remarked that the solutions of field equations represent just a small part of the information that can be extracted out of a field theory. Other relevant information is encoded in the field symmetries; the knowledge on how fields are dragged along infinitesimal generators of symmetries is hence a further fundamental detail.

Luckily enough, the two main ingredients which have to enter into a satisfactory definition of conserved quantities, namely the field equations content and the symmetry information, can be joined together into a unique structure. Field equations are indeed described via the Euler–Lagrange morphism  $e(L)$  which turns out to be a differential form on (some suitable jet prolongation of) the configuration bundle of the theory. The Euler–Lagrange morphism can be contracted with the Lie derivatives  $\mathcal{L}_\xi y$  of the fields  $y$  with respect to infinitesimal generators  $\xi$  of symmetries (which define vertical vector fields). The resulting object  $L' := - \langle e(L) | \mathcal{L}_\xi y \rangle$  turns out to be a horizontal form which can be interpreted as a new Lagrangian. From now on we shall refer to  $L'$  as *the variational Lagrangian* (associated to  $L$  and  $\xi$ ). The variational Lagrangian, for the information it encodes, is then in a good position to represent the fundamental

object out of which we derive conserved quantities: being  $L'$  a Lagrangian it can in fact be handled by means of the powerful tools of the Calculus of Variations in jet bundles.

Our recipe to define *the variation* of conserved quantities will be thence developed in two steps which are both canonically and algorithmically well-defined at the jet bundle level. The first one is nothing but the *first variational formula* applied to the variational Lagrangian. The variation of  $L'$ , through a well-known integration by parts procedure, splits canonically into the Euler–Lagrange part (which vanishes because of symmetry properties) plus a pure divergence term  $\text{Div } \mathcal{F}$ . The so-called Poincaré–Cartan morphism  $\mathcal{F}$  which (non uniquely but in a sense canonically) enters this divergence can be expanded as a linear combination of the coefficients of the vector field  $\xi$  (generating the flow of symmetries) together with their covariant derivatives up to an appropriate (finite) order. The second step consists in implementing the so-called Spencer cohomology [38, 42] through repeated integrations by parts with respect to these covariant derivatives. In this way we end up with a  $(m - 2)$  form  $\mathcal{U}$ , called *the potential*, where  $m$  is the dimension of spacetime, which we choose to be a canonical representative, at the bundle level, suited to define the variations of conserved quantities. These latter quantities are indeed obtained by integrating the pull-back of  $\mathcal{U}$  along solutions on the appropriate  $(m - 2)$ -dimensional domains of spacetime. We point out that the possibility of selecting a canonical representative for the potential  $\mathcal{U}$  is clearly an essential task, since different representatives may lead to different notions of conserved quantities.

Notice that only the *variation*  $\delta\mathcal{Q}$  of a conserved quantity, rather than the conserved quantity  $\mathcal{Q}$  itself, is here defined. However, one can argue that  $\delta\mathcal{Q}$  rather than  $\mathcal{Q}$  is the truly fundamental object. Physically speaking, in fact, conserved quantities are not absolute, since only differences of physical observables are endowed with a direct physical meaning. Therefore one has to somehow fix a “reference point” (e.g. a background) to calculate them. However, in the absence of a linear structure on the space of fields there is no canonical choice for such a “reference point”. Moreover, the formal on-shell integration of  $\delta\mathcal{Q}$  depends on how we move along a curve in the space  $\mathcal{E}$  of solutions. Starting from a given solution  $\varphi_0$ , different deformations  $\delta$  correspond to different paths in  $\mathcal{E}$  passing through the same point  $\varphi_0$ , so that  $\delta\mathcal{Q}$  measures the variation of  $\mathcal{Q}$  when we move from  $\varphi_0$  to a nearby solution in a given direction. Roughly speaking the choice of such a direction corresponds to a control mode of the fields at the boundary which is based on physical grounds (e.g. micro-canonical or grand-canonical ensembles) and it is related to the asymptotic behaviour of specific solutions and to the specific physical observable one is aimed to determine (see, e.g., [4, 9, 16, 19, 40, 50, 53]). We then believe that, inside a formalism which we want to be as general as possible, both physical and mathematical reasons suggest to us that only variations  $\delta\mathcal{Q}$  should be reasonably considered.

Finally, we point out another property that a viable definition of conserved quantities has to fulfill (and according to which we have developed the present

formalism). Conserved quantities are mathematically defined objects which are built out of fields with the purpose of extracting information about the physical properties of all solutions; they represent indeed physical observables such as mass, energy, momentum, angular momentum or gauge charges. For this reason the mathematical definition of conserved quantities, whatever formalism we implement to obtain it, must eventually get rid of the mathematical structure we started from and, in the meanwhile, it must not depend on the specific configuration variables we have selected to describe the solution itself. Let us better explain this “philosophical” viewpoint by considering as an example the theory of gravitation. It is well-known that there exist different Lagrangian formulations of gravity (e.g. purely metric [47], metric affine [24], purely affine [23], tetrad formulations [52], Ashtekar variables [7], Chern–Simons formulation [1] and so on); these are described by profoundly different Lagrangians which in many cases are “equivalent” in the sense that they determine equivalent field equations even if they do not merely differ for the addition of divergence terms. From a geometric viewpoint these are really different theories since they are based on different configuration spaces and involve different dynamical fields. Nevertheless, all the aforementioned theories, under suitable regularity conditions, are equivalent on shell (i.e. they generate essentially identical or at least isomorphic spaces of solutions, possibly under appropriate gauge reductions). Basically, there exists a rule (which can be many-to-one) which maps a solution of one theory into a solution of another.<sup>1</sup> For each different formulation of gravity we expect then a different setting for the definition of conserved quantities. Nevertheless it is physically desirable that all different definitions of conserved quantities turn eventually out to coincide on-shell; otherwise we may risk to obtain a paradoxical conceptual result. Indeed, if on-shell equivalent gravitational theories admit solutions that correspond to the same spacetime, all physically meaningful observables must be related to the spacetime itself and should not depend on which particular variable we have used to describe it. For example, the energy enclosed in a (bounded) region surrounding a black hole solution must depend solely on the solution and not on the specific configuration variable (e.g. metric, tetrad or connection) we initially choose to describe it.

The formalism developed in this paper partially avoids such a possible paradox. In fact, it is based on the variational Lagrangian which is obtained via contraction of the equations of motion with the Lie derivatives of fields. If any rule exists to unambiguously map the equations of motion and the Lie derivatives of one theory into the corresponding objects of another theory, then there exists, of course, an unambiguous correspondence between the variational Lagrangians, too. Accordingly, also the conserved quantities of the first theory will be mapped into the conserved quantities of the second, so that, on-shell, they will eventually give rise in both approaches to the same numerical value for each conserved integral.

However, we have to stress that there exist Lagrangian theories which ad-

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<sup>1</sup>For example purely metric, metric affine and purely affine theories are all related by a generalized Legendre transformation and the quoted map among solutions is just given by the Legendre map [54]

mit different symmetry groups even if they are equivalent as far as their field equations content is concerned. For instance, purely metric, metric affine and purely affine theories are natural formulations of gravity and they all admit the group of spacetime diffeomorphisms as the “natural” symmetry group. On the other hand tetrad or Chern–Simons formulations of General Relativity are truly gauge natural formulations so that they admit a much larger group of symmetries, which, besides spacetime diffeomorphisms, encompasses also gauge transformations.

In general, in gauge natural theories it happens that pure spacetime transformations cannot be globally isolated (in non trivial topologies) from gauge transformations. This fact reverberates then into the indeterminacy in defining the Lie derivative of gauge natural objects with respect to arbitrary spacetime vector fields (while such an indeterminacy does not occur for natural objects).<sup>2</sup> Many lifts of the same vector field can be thence chosen: they originate different Lie derivatives and, accordingly, different variational Lagrangians and different conserved quantities. This fact raises a very interesting issue about symmetries (which we will face up in the last section). Namely, albeit we have a correspondence between equations of motion we, *a priori*, lack a strict correspondence between symmetries. Nevertheless we claim that, *a posteriori*, also the latter correspondence can be gained in our approach and such a goal is simply obtained by imposing the equivalence of the variational Lagrangians. In this way a preferred lift of spacetime vector fields can be ruled out in gauge natural theories: it is the lift which restores the naturality in the gauge natural variational Lagrangian.

The present paper is organized as follows. In section 2, in order to make the paper self-contained, we shortly review the geometric framework of natural and gauge natural theories. In section 3 we present the theoretical formulation of the “two steps procedure” which leads to the definition of the variation of conserved quantities. The result achieved here is tested for the purely metric formulation of General Relativity (in section 4) and for the gauge natural tetrad formulation of General Relativity (in section 5). No discrepancies are observed between the two different formulations owing to the specific form of the variational Lagrangians. The last section (section 6) is finally devoted to the analysis of Einstein–Cartan theory in the gauge natural framework. This theory is a generalization of Einstein’s theory and, in vacuum, it becomes on-shell equivalent to it. We have chosen to investigate this theory since it clearly features the indeterminacy we mentioned above when defining conserved quantities in gauge natural theories with respect to spacetime vector fields. It will become however clear how this indeterminacy can be *a posteriori* eliminated by restoring naturality in the gauge natural variational Lagrangian. This procedure selects in the

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<sup>2</sup>E.g., spinors, which are truly gauge natural objects cannot be classically Lie-dragged along generic spacetime vectors, but a classical notion of Lie derivative exists along Killing vectors, i.e. infinitesimal symmetries of the spacetime metric. A general definition of Lie derivative of spinors can be achieved *only* in the gauge natural formalism (see [27, 41, 59]).

specific example the so-called *generalized Kosmann lift* as the preferred lift to deal with. As a by-product we gain some new insight on the mathematical justification of the Kosmann lift previously introduced in [27, 41, 59] in the domain of gauge natural theories as a generalization of an *ad hoc* procedure introduced by Kosmann in [58] to Lie drag spinors.

## 2 Natural and gauge natural theories

Natural theories geometrically formalize the physical principle of general covariance. According to [35, 56] we say that a field theory is *natural* when:

- A: the configuration bundle is natural;
- B: the Lagrangian describing the theory is natural.

Item A means that spacetime diffeomorphisms can be functorially lifted to the configuration bundle  $Y$  (i.e. the space where the dynamical fields  $\varphi$  take their values). Roughly speaking, we know how fields transform under changes of coordinates in spacetime. As a consequence of naturality, for each spacetime vector field  $\xi$  there exists a canonical lift  $\hat{\xi}$  on  $Y$  (which projects onto  $\xi$ ). Using the natural lift  $\hat{\xi}$  it is then meaningful to consider Lie derivatives of fields with respect to spacetime vector fields by setting  $\mathcal{L}_\xi \varphi := \mathcal{L}_{\hat{\xi}} \varphi = T\varphi \circ \xi - \hat{\xi} \circ \varphi$ .

Item B means that all spacetime transformations, once lifted on  $Y$ , are symmetries. Accordingly, each lift  $\hat{\xi}$  is an infinitesimal generator of symmetries.

Obviously all theories based on diffeomorphism invariant Lagrangians which depend on tensor fields, tensor densities and/or linear connections are natural theories according to the given definition.

In order to encompass into a unique geometric formalism theories admitting both diffeomorphism invariance as well as gauge symmetries one is led to introduce *gauge natural theories*. In gauge natural theories we assume that there exists a principal bundle  $(P, M, p; G)$ , called *the structure bundle*, where all information concerning symmetries is encoded; gauge natural theories admit the group  $\text{Aut}(P)$  of all automorphisms of the structure bundle as group of symmetries (pure gauge symmetries correspond to vertical automorphisms). This amounts to say that a theory is gauge natural (see [27, 33, 56] for a deeper geometric insight) when:

- A: the configuration bundle is gauge natural;
- B: the Lagrangian describing the theory is gauge natural;
- C: a linear connection  $\Gamma$  and a principal connection  $A$  on the structure bundle can be built out of the dynamical fields.

Item A is a pure geometric requirement: it means that the automorphisms of the structure bundle  $P$  functorially induce automorphisms on the configuration bundle  $Y$ . Accordingly, projectable vector fields  $\Xi_P$  on  $P$  canonically induce vector fields  $\Xi$  on  $Y$ . Item B is instead of dynamical nature since it implies

that all such induced vector fields  $\Xi$  are infinitesimal generators of symmetries. Therefore we should be somehow able to associate conservation laws to each of them leading to quantities which are eventually physically interpretable as observables. In doing that we shall need the Lie derivatives of fields with respect to vector fields  $\Xi_P$ , which are defined by setting

$$\mathcal{L}_{\Xi_P}\varphi := \mathcal{L}_{\Xi}\varphi = T\varphi \circ \xi - \Xi \circ \varphi \quad (1)$$

where  $\xi$  denotes the projection of  $\Xi_P$  onto spacetime  $M$ . Notice however that the group  $\text{Diff}(M)$  is not canonically embedded into  $\text{Aut}(P)$ . We know how fields transform as a consequence of a transformation in  $P$  but we do not know how fields transform under change of coordinates in spacetime. In other words Lie derivatives with respect to spacetime vector field cannot be, at least a priori, defined for a generic  $\xi$ .

Finally, item C is a technical requirement: the two *dynamical* connections  $\Gamma$  and  $A$  are the mathematical tools which are necessary to provide covariance at each step of the geometric formalism used to generate conserved quantities.

**Remark 2.1** We stress that despite in gauge natural theories there exists no natural way to define the action of diffeomorphisms on the dynamical fields there exist, however, *many* global (but not canonical) ways to lift spacetime vector fields up to the structure bundle. One of this, for example, is *the horizontal lift* defined through the dynamical connection  $A$ . Even if this lift apparently seems to be the most “natural” way to define the lift of vector fields, it has been nevertheless shown [3] that it does not lead to physically acceptable values for conserved quantities in physical applications so that eventually one has to resort to some other lift. In the applications we shall deal with, it will be shown that the generalized Kosmann lift is the most “natural” one; see section 6. ■

**Remark 2.2** We remark that, as far as the geometric formalism for conserved quantities is concerned, we are not so much interested in the Lagrangian but rather in the variational equations ensuing from it. Therefore we could have weakened item B by just requiring that the dynamical equations describing the physical system are (gauge) covariant. Nevertheless, starting from a set of covariant equations of variational nature it is always possible, at least in principle, to build out a covariant family of Lagrangians depending on a (dynamical) background; see [39] (see also [3, 12] where covariant Lagrangians for Chern–Simons theories are exhibited). In the sequel we shall implicitly assume that any one of these Lagrangians has been already selected, even if we shall not be interested into its explicit form. ■

### 3 Conserved Quantities from the Equations of Motions

Let us consider a gauge natural theory geometrically described through a gauge natural bundle  $(Y, M, \pi)$  and dynamically defined by a  $k$ -order Lagrangian  $L$  :

$J^k Y \longrightarrow \Lambda^m(M)$ . In terms of fibered coordinates  $(x^\mu, y^i)$  on  $Y$  we locally have  $L = \mathbb{L}(j^k y) ds$  where  $\mathbb{L}$  is the Lagrangian density,  $ds = dx^1 \wedge \dots \wedge dx^m$  is the standard (local) volume form on  $M$  and  $(j^k y)$  stands for  $(y^i, y_\mu^i, \dots, y_{\mu_1 \dots \mu_k}^i)$ , i.e. the set of partial derivatives of fields up to order  $k$  included. We shall denote by  $J^k Y$  the  $k$  jet prolongation of  $Y$ , by  $V(J^k Y)$  its vertical tangent bundle and by  $V^*(J^k Y)$  the vector bundle dual to  $V(J^k Y)$ . Notation and definitions in this section follow closely [33], to which we refer the reader for a full treatment and further details.

A *variation* is a vertical vector field  $X$  on the configuration bundle, which can be locally described as  $X = X^i \frac{\partial}{\partial y^i} = \delta y^i \frac{\partial}{\partial y^i}$ ; physically speaking it describes a one-parameter deformation of the dynamical fields. Accordingly, we can consider the variation  $\delta_X L$  of the Lagrangian along the flow of (the prolongation of) the vector field  $X$ . It is well-known (see [35, 55, 65]) that each Lagrangian  $L$  induces a unique (global) morphism, called *the Euler-Lagrange morphism*

$$e(L) : J^{2k} Y \longrightarrow V^*(Y) \otimes \Lambda^m(M) \quad (2)$$

together with a family of (global) morphisms (which depend on the Lagrangian and possibly on a connection  $\Gamma$  on  $M$ ) called *Poincaré-Cartan morphisms*

$$\mathcal{F}(L, \Gamma) : J^{2k-1} Y \longrightarrow V^*(J^{k-1} Y) \otimes \Lambda^{m-1}(M) \quad (3)$$

The Euler-Lagrange morphism and the Poincaré-Cartan morphisms are in fact defined so that the so-called *first-variation formula* holds for any deformation  $X$  on  $Y$ :

$$\delta_X L = \langle e(L) | X \rangle + \text{Div} \langle \mathcal{F}(L, \Gamma) | j^{k-1} X \rangle \quad (4)$$

where  $\langle | \rangle$  denotes the canonical pairing between differential forms and vector fields (in our case between elements of  $V^*(J^h Y)$  and elements of  $V(J^h Y)$ ). The *formal divergence operator on forms* is defined by

$$\text{Div}(f) \circ j^{k+1} \varphi = d(f \circ j^k \varphi), \quad f : J^k Y \longrightarrow \Lambda(M) \quad (5)$$

$d(\cdot)$  being the exterior differential operator on forms and  $\Lambda(M) \equiv \oplus_k \Lambda^k(M)$  denoting the bundle of forms over  $M$ . The Poincaré-Cartan morphisms are uniquely defined only for  $k = 0$  or  $k = 1$ ; for  $k = 2$  they are not unique but still there is a canonical choice which is independent on any connection  $\Gamma$ . For  $k > 2$  the Poincaré-Cartan morphisms strongly depend on the choice of a linear connection; see [34]. Nevertheless, in gauge natural theories the existence of a dynamical linear connection  $\Gamma$  is axiomatically required (see item C of the definition); in this case we shall implicitly assume that the Poincaré-Cartan morphism entering the first variation formula (for  $k > 2$ ) is the one induced by this preferred connection  $\Gamma$ . For this reason, from now on we shall omit to indicate in the notation the dependence on  $\Gamma$  of  $\mathcal{F}(L, \Gamma)$  and we shall simply write  $\mathcal{F}(L)$ .

The Euler-Lagrange morphism (2), which can be locally written as

$$e(L) = e_i(j^{2k} y) dy^i \otimes ds \quad (6)$$



encodes the information relative to the equations of motion for the dynamical fields. A *critical section* (or a solution) is a section  $\varphi : M \longrightarrow Y$ , locally described as  $\varphi : x^\mu \mapsto (x^\mu, y^i = \varphi^i(x))$ , the prolongation of which belongs to the kernel of the Euler–Lagrange morphism, i.e.:

$$e(L) \circ j^{2k}\varphi = 0 \implies e_k(\varphi^i, d_\mu \varphi^i, \dots, d_{\mu_1 \dots \mu_{2k}} \varphi^i) = 0 \quad (7)$$

Now, let us denote by  $\Xi_P$  a projectable vector field on the relevant principal bundle  $(P, M, p)$  of the theory. It locally reads as  $\Xi_P = \xi^\mu \partial_\mu + \xi^A \rho_A$  (where  $\rho_A$  is a local basis for right invariant vertical vector fields on  $P$ ). It canonically induces a vector field  $\Xi$  on the configuration bundle which is, by the very definition of gauge natural theory (see the previous section), an infinitesimal generator of symmetries. Through the vector field  $\Xi$  we can define the (formal) Lie derivative  $\mathcal{L}_{\Xi} y$  of the fields. Since  $\mathcal{L}_{\Xi} y : J^1 Y \longrightarrow V(Y)$  takes value into the vertical bundle of  $Y$  it is meaningful to consider the contraction

$$L'(L, \Xi) = - \langle e(L) | \mathcal{L}_{\Xi} y \rangle : J^{2k} Y \longrightarrow \Lambda^m(M) \quad (8)$$

which defines a horizontal form on the bundle  $J^{2k} Y$  which we shall call, from now on, *the variational Lagrangian*.

Since we are assuming the configuration bundle to be a gauge natural bundle the Lie derivative  $\mathcal{L}_{\Xi} y$  entering into the definition of the variational Lagrangian can be written as a linear combination of symmetrized covariant derivatives with respect to the dynamical connections  $(\Gamma, A)$ ; see [28, 38]. Thereby expression (8) can be locally written as

$$\begin{aligned} L'(L, \Xi) = & \left\{ W_\mu \xi^\mu + W_\mu^{\rho_1} \nabla_{\rho_1} \xi^\mu + \dots + W_\mu^{\rho_1 \dots \rho_r} \nabla_{(\rho_1 \dots \rho_r)} \xi^\mu \right. \\ & \left. + W_A \xi^A + W_A^{\rho_1} \nabla_{\rho_1} \xi^A + \dots + W_A^{\rho_1 \dots \rho_s} \nabla_{(\rho_1 \dots \rho_s)} \xi^A \right\} ds \end{aligned} \quad (9)$$

where  $W_\mu, W_\mu^{\rho_1}, \dots, W_\mu^{\rho_1 \dots \rho_s}, W_A, W_A^{\rho_1}, \dots, W_A^{\rho_1 \dots \rho_r}$  are *tensor densities* with respect to automorphisms of the structure bundle (the pair  $(r, s)$  is called *the order* of the gauge natural bundle; see [56]); here and in the sequel round brackets around indices denote symmetrization.

Whenever we have such a linear combination we can perform covariant integration by parts to obtain for  $L'(L, \Xi)$  an equivalent canonical expansion under the form:

$$L'(L, \Xi) = \mathcal{B}(L, \Xi) + \text{Div } \tilde{\mathcal{E}}(L, \Xi) \quad (10)$$

where the quantity  $\tilde{\mathcal{E}}(L, \Xi)$  is usually called the *reduced current* and vanishes on-shell, while the quantity  $\mathcal{B}(L, \Xi)$  is linear in  $\Xi$  and it turns out to be identically vanishing along any section (see [35, 38]). The identities

$$\mathcal{B}(L, \Xi) = 0 \quad (11)$$

are called *generalized Bianchi identities* (indeed they reduce to the usual Bianchi identities in General Relativity and in gauge theories). Hence equation (10) gives rise to a conservation law:

$$\text{Div } \tilde{\mathcal{E}}(L, \Xi) = L'(L, \Xi) \simeq 0 \quad (12)$$

where  $\simeq$  denotes equality on-shell. Nevertheless it is clear that (12) is a *trivial* conservation law owing to the fact that the reduced current  $\tilde{\mathcal{E}}$ , which is built out of the coefficients (9) of the Euler–Lagrange morphism together with their covariant derivatives, is, as we said, vanishing on-shell. Hence equation (12) is of no utility by itself to describe physical properties of any given solution. Nevertheless, the variational Lagrangian becomes the starting point to define algorithmically the quantities we are interested in. The formalism basically develops in two steps.

**First step: integration by parts with respect to  $X$ .**

Let us consider a vertical vector field  $X$  on the configuration bundle, locally given as  $X = X^i \frac{\partial}{\partial y^i} = \delta y^i \frac{\partial}{\partial y^i}$ . By treating (8) as a new Lagrangian we can consider the variation  $\delta_X L'$  and, accordingly, we can make again use of the first variational formula (4). Notice that  $L'$  in general depends on the fields  $y^i$  together with their derivatives up to some order  $h \leq 2k$ , but it also depends on the variables  $\xi^\mu$  and  $\xi^A$  together with their derivatives up to the orders  $r$  and  $s$ , respectively; see (9).<sup>3</sup> In the variation  $\delta_X L'$  the components  $\xi^\mu$  and  $\xi^A$  are independent on fields and thereby are kept fixed, i.e.  $\delta_X \xi^\mu = \delta_X \xi^A = 0$ . In the sequel we shall also be interested in the case in which the components  $\xi^A$ , via some suitable “lift”, are built out of the dynamical fields  $y^i$  and the coefficients  $\xi^\mu$  together with their derivatives up to some fixed order, i.e.  $\xi^A = \xi^A(j^a y, j^b \xi^\mu)$ . When such a dependence is inserted back into (8) we end up with a variational Lagrangian which depends only on the dynamical fields and the components  $\xi^\mu$ . Thereby the whole formalism we are going to develop will simply work by setting  $\xi^A = 0$ .

The first variation formula applied to the Lagrangian  $L'$  reads now as follows:

$$\begin{aligned} \delta_X L' &= \langle e(L') | X \rangle + \text{Div } \mathcal{F}(L', X) \\ &= \text{Div } \mathcal{F}(L', X) \end{aligned} \quad (13)$$

where  $\mathcal{F}(L', X)$  stands for  $\langle \mathcal{F}(L') | j^p X \rangle$  for some suitable  $p$ . The latter equality in (13) is due to the fact that, being the Lagrangian  $L'$  a total divergence (see (12)) its associated Euler–Lagrange morphism  $\langle e(L') | X \rangle = \langle e(\text{Div } \tilde{\mathcal{E}}) | X \rangle$  vanishes identically. Moreover the variation of (12) turns out to be a pure divergence, i.e.:

$$\delta_X L' = \text{Div } \delta_X \tilde{\mathcal{E}}(L, \Xi) \quad (14)$$

Hence, comparing (13) with (14) we obtain a *strong* conservation law (i.e. a conservation law which holds true off-shell):

$$\text{Div } \left\{ \mathcal{F}(L', X) - \delta_X \tilde{\mathcal{E}}(L, \Xi) \right\} = 0 \quad (15)$$

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<sup>3</sup>The inequality  $h \leq 2k$  is due to the fact that a Lagrangian of order  $k$  can give rise to field equations of order lower than  $2k$ ; see, e.g. General Relativity or Lovelock metric theories [18] where  $k = 2$  while field equations are second order only.

In particular, if the variation  $X$  is a solution of the linearized field equations, i.e. it is tangent to the space of solutions, then  $\delta_X \tilde{\mathcal{E}}(L, \Xi) = 0$  and we obtain a *weak* conservation law:

$$\text{Div } \mathcal{F}(L', X) \simeq 0 \quad (16)$$

**Second step: integration by parts with respect to  $\Xi$ .**

We shall now show that the  $(m-1)$  form  $\mathcal{F}(L', X) - \delta_X \tilde{\mathcal{E}}(L, \Xi)$  is not only closed but it is also exact (off-shell) and we shall explicitly exhibit a global representative for its potential. This is a result which does not depend on the topology of spacetime and it does not depend on any given solution. Indeed all calculations will be performed on some jet prolongation of the configuration bundle  $Y$ . Only at the end of the calculations we shall pull-back the results along sections of the appropriate jet bundle obtaining in this way differential forms on the base manifold. Moreover the global character of the results we shall obtain will follow directly from the fact that the algorithm developed fulfills the covariance property at each step of its construction.

First of all let us consider the morphism  $\mathcal{F}(L', X)$ . Since  $L'$  depends on  $j^h y$ ,  $j^r \xi^\mu$  and  $j^s \xi^A$ , with  $h \leq 2k$ , and the expression in the right hand side of (13) is obtained, in general, through  $h$  integration by parts, the quantity  $\mathcal{F}(L', X)$  depends linearly on the independent components  $\xi^\mu$ ,  $\xi^A$  together with their derivatives up to the orders, respectively,  $h+r-1$  and  $h+s-1$ . Therefore it can be written as

$$\begin{aligned} \mathcal{F}(L', X) = & \left\{ \mathcal{F}_\mu^\rho \xi^\mu + \mathcal{F}_\mu^{\rho\rho_1} \nabla_{\rho_1} \xi^\mu + \dots + \mathcal{F}_\mu^{\rho\rho_1 \dots \rho_{h+r-1}} \nabla_{(\rho_1 \dots \rho_{h+r-1})} \xi^\mu \right. \\ & \left. + \mathcal{F}_A^\rho \xi^A + \mathcal{F}_A^{\rho\rho_1} \nabla_{\rho_1} \xi^A + \dots + \mathcal{F}_A^{\rho\rho_1 \dots \rho_{h+s-1}} \nabla_{(\rho_1 \dots \rho_{h+s-1})} \xi^A \right\} ds_\rho \end{aligned} \quad (17)$$

where  $ds_\rho = \partial_\rho \rfloor ds$ . The coefficients in (17) are *tensor densities* which depend on the dynamical fields  $y^i$  together with their variations  $X^i = \delta y^i$  up to some (finite) order. Notice that covariant derivatives can be defined with respect to the dynamical connections (see item C of the definition). Notice also that the coefficients  $\mathcal{F}_\mu^{\rho\rho_1 \dots \rho_p}$  and  $\mathcal{F}_A^{\rho\rho_1 \dots \rho_q}$  are of course symmetric in the indices  $\rho_i$  but not with respect to the whole set of upper indices. However, whenever we have a linear combination of the kind (17) we can perform covariant integration by parts to obtain for the same quantity an equivalent linear expansion the coefficients of which are all symmetric with respect to upper indices, while the integrated terms are all pushed into a formal divergence. In this way expression (17) can be recasted as follows: <sup>4</sup>

$$\mathcal{F}(L', X) = \tilde{\mathcal{F}}(L', X) + \text{Div } \mathcal{U}(L', X) \quad (20)$$

<sup>4</sup> For the sake of completeness of this paper we give the explicit formula for the decomposition in the case  $h+r-1=2$ ,  $s=0$  and  $\Gamma^\alpha_{\mu\nu}$  symmetric. We have in this case:

$$\begin{aligned} \tilde{\mathcal{F}}_\mu^\alpha &= \mathcal{F}_\mu^\alpha + f_\mu^\alpha - \nabla_\beta (\mathcal{F}_\mu^{[\alpha\beta]} + f_\mu^{[\alpha\beta]}) \\ \tilde{\mathcal{F}}_\mu^{(\alpha\beta)} &= \mathcal{F}_\mu^{(\alpha\beta)} + f_\mu^{(\alpha\beta)} \\ \tilde{\mathcal{F}}_\mu^{(\alpha\beta\gamma)} &= \mathcal{F}_\mu^{(\alpha\beta\gamma)} \end{aligned} \quad (18)$$

$$\begin{aligned}
&= \left\{ \tilde{\mathcal{F}}_\mu^\rho \xi^\mu + \tilde{\mathcal{F}}_\mu^{(\rho\rho_1)} \nabla_{\rho_1} \xi^\mu + \dots + \tilde{\mathcal{F}}_\mu^{(\rho\rho_1 \dots \rho_{h+r-1})} \nabla_{(\rho_1 \dots \rho_{h+r-1})} \xi^\mu \right. \\
&\quad \left. + \tilde{\mathcal{F}}_A^\rho \xi^A + \tilde{\mathcal{F}}_A^{(\rho\rho_1)} \nabla_{\rho_1} \xi^A + \dots + \tilde{\mathcal{F}}_A^{(\rho\rho_1 \dots \rho_{h+s-1})} \nabla_{(\rho_1 \dots \rho_{h+s-1})} \xi^A \right\} ds_\rho \\
&\quad + d_\sigma \mathcal{U}^{[\rho\sigma]} ds_\rho
\end{aligned}$$

Here and in the sequel square brackets around indices denote skew-symmetrization. (For the general setting of the above decomposition for any pair  $(r, s)$  we refer the reader to [38]). We only stress that this kind of integration by parts is a well-defined global operation in the bundle framework. Globality is indeed assured by the direct use of covariant derivatives. Uniqueness of  $\tilde{\mathcal{F}}$  instead descends from its symmetry properties and from the fact that  $\text{Div}$  is a nilpotent operator:  $\text{Div} \circ \text{Div} = 0$ . Therefore, it can easily be checked that the term  $\tilde{\mathcal{F}}(L', X)$  in the right hand side of (20) coincides with the variation  $\delta_X \tilde{\mathcal{E}}(L, \Xi)$  of the reduced current and it is hence vanishing on-shell. Indeed, inserting (20) into (15) and taking  $\text{Div} \circ \text{Div} = 0$  into account we have:

$$\text{Div} \left\{ \tilde{\mathcal{F}}(L', X) - \delta_X \tilde{\mathcal{E}}(L, \Xi) \right\} = 0 \quad (21)$$

Since the components  $\xi^\mu$  and  $\xi^A$  are arbitrary and since both terms  $\tilde{\mathcal{F}}(L', X)$  and  $\delta_X \tilde{\mathcal{E}}(L, \Xi)$  are, by construction, symmetric in the upper indices, it is easy to verify that (21) implies  $\tilde{\mathcal{F}}(L', X) = \delta_X \tilde{\mathcal{E}}(L, \Xi)$ .

Also the potential  $\mathcal{U}$  features an expression of the kind:

$$\begin{aligned}
\mathcal{U}^{[\rho\sigma]} &= \mathcal{U}_\alpha^{[\rho\sigma]} \xi^\alpha + \dots + \mathcal{U}_\mu^{[\rho\sigma](\rho_1 \dots \rho_{h+r-2})} \nabla_{(\rho_1 \dots \rho_{h+r-2})} \xi^\mu \\
&\quad + \mathcal{U}_A^{[\rho\sigma]} \xi^A + \dots + \mathcal{U}_A^{[\rho\sigma](\rho_1 \dots \rho_{h+s-2})} \nabla_{(\rho_1 \dots \rho_{h+s-2})} \xi^A
\end{aligned} \quad (22)$$

which is clearly not unique but only defined modulo closed forms. Nevertheless (see [38]) there exists a unique representative the coefficients of which have the maximal symmetry property:  $\mathcal{U}^{[\rho\sigma\rho_1] \dots \rho_p} = 0$ . From now on we shall select this representative which is, from a mathematical viewpoint, the *canonical* one. The physical viability of this choice will be tested in applications.

Let us summarize. Starting from the variational Lagrangian (8) we have obtained a current:

$$\mathcal{F}(L', X) = \delta_X \tilde{\mathcal{E}}(L, \Xi) + \text{Div} \mathcal{U}(L', X) \quad (23)$$

which is conserved on-shell (see (16)) and it is also exact on-shell if  $X$  is a solution of the linearized field equations, since in this case the term  $\delta_X \tilde{\mathcal{E}}(L, \Xi)$

with  $f_\mu^\alpha = 1/3 \mathcal{F}_\gamma^{[\sigma\beta]\alpha} R_{\mu\sigma\beta}^\gamma$ ,  $f_\mu^{\alpha\beta} = -4/3 \nabla_\sigma \mathcal{F}_\mu^{[\alpha\sigma]\beta}$  and

$$\mathcal{U}^{\alpha\beta} = \left\{ \mathcal{F}_\mu^{[\alpha\beta]} - 2/3 \nabla_\sigma \mathcal{F}_\mu^{[\alpha\beta]\sigma} \right\} \xi^\mu + 4/3 \mathcal{F}_\mu^{[\alpha\beta]\sigma} \nabla_\sigma \xi^\mu \quad (19)$$

The generalization to  $s \neq 0$  is easily obtained replacing lower Greek indices with Latin indices, replacing the Riemann tensor with the field strength  $F$  and enlarging the covariant derivatives to act on both internal and spacetime indices.

in (23) is vanishing (notice, instead, that the current  $\mathcal{F}(L', X) - \delta_X \tilde{\mathcal{E}}(L, \Xi)$  is exact also off-shell).

Let us now consider a section  $\varphi : M \longrightarrow Y$ . Its prolongation  $j\varphi$  to a suitable (finite) order can be used to pull-back expression (23) onto the base manifold. We shall denote by  $\mathcal{F}(X, \varphi)$ ,  $\tilde{\mathcal{E}}(\Xi, \varphi)$  and  $\mathcal{U}(X, \varphi)$  the pull-backs on  $M$  of the relevant quantities in (23) (from now on the indication  $L'$  between the round brackets will be omitted for the sake of simplicity). Given a hypersurface  $D$  in spacetime we define the *variation*  $\delta_X \mathcal{Q}_D(\Xi, \varphi)$  of the conserved quantity  $\mathcal{Q}_D(\Xi, \varphi)$ , relative to the set  $(\Xi, \varphi, D)$  as follows:

$$\delta_X \mathcal{Q}_D(\Xi, \varphi) := \int_D \mathcal{F}(X, \varphi) = \int_D \delta_X \tilde{\mathcal{E}}(\Xi, \varphi) + \int_{\partial D} \mathcal{U}(X, \varphi) \quad (24)$$

$$\simeq \int_{\partial D} \mathcal{U}(X, \varphi) \quad (25)$$

where the last equality holds true whenever  $\varphi$  is a solution of field equations and  $X$  is tangent to the space of the solutions, i.e. it describes a one-parameter deformation of  $\varphi$  along nearby solutions. In particular if the region  $D$  is a (portion of a) Cauchy surface and the vector field  $\xi$  is transverse to  $D$  we identify  $\delta_X \mathcal{Q}_D(\Xi, \varphi)$  in (24) with the variation of the Hamiltonian  $\delta_X \mathcal{H}_D(\Xi, \varphi)$ . We stress that, for the time being, this is just a definition. Nevertheless the name Hamiltonian will be justified by the applications we shall explicitly consider. Indeed it will turn out that  $\delta_X \mathcal{H}_D(\Xi, \varphi)$  physically describes the evolution of the fields along the flow of the vector  $\xi$ .

**Remark 3.1** We remark that, even though we have assumed a gauge natural Lagrangian as the starting point for our framework, the whole theory is in fact developed from the Euler–Lagrange morphism (8). The Lagrangian is necessary, a priori, only in defining the dynamics via its equations of motion. This implies that it is of no interest whatsoever Lagrangian representative we choose in the cohomology class  $[L]$  the elements of which differ from each other only by the addition of divergences. Indeed all the representatives inside the class  $[L]$  give rise to the same equations. Even more, we can relax the hypothesis that the Lagrangian be gauge natural since only the equations of motion are required to be gauge invariant as well as diffeomorphism invariant. Hence, our formalism allows to encompass also more general classes of field theories, e.g. Chern–Simons theories. ■

**Remark 3.2** Notice that, through a direct application of the Noether theorem, starting from the Lagrangian  $L$  and from the infinitesimal generator of symmetries  $\Xi$ , one is able to algorithmically construct a conserved Noether current  $\mathcal{E}(L, \Xi)$  and to exhibit a superpotential  $\mathcal{V}(L, \Xi)$  such that the following two identities hold:

$$\text{Div } \mathcal{E}(L, \Xi) = L'(L, \Xi) \simeq 0 \quad (26)$$

$$\mathcal{E}(L, \Xi) = \tilde{\mathcal{E}}(L, \Xi) + \text{Div } \mathcal{V}(L, \Xi) \quad (27)$$

where  $\tilde{\mathcal{E}}(L, \Xi)$  is the same reduced current appearing in (23) (see [28, 35] for details). By taking the variation of (26) and taking (13) into account we then get:

$$\text{Div } \delta_X \mathcal{E}(L, \Xi) = \text{Div } \mathcal{F}(L', X) \quad (28)$$

so that  $\delta_X \mathcal{E}(L, \Xi)$  and  $\mathcal{F}(L', X)$  differ from each other for a form which (at least locally) is exact:

$$\begin{aligned} \mathcal{F}(L', X) &= \delta_X \mathcal{E}(L, \Xi) + \text{Div } \mathcal{C}(L, \Xi, X) \\ &\stackrel{(27)}{=} \delta_X \tilde{\mathcal{E}}(L, \Xi) + \text{Div } \{ \delta_X \mathcal{V}(L, \Xi) + \mathcal{C}(L, \Xi, X) \} \end{aligned} \quad (29)$$

Notice now that the divergence  $\text{Div } \mathcal{C}$  cannot be vanishing in general. Indeed, if  $\text{Div } \mathcal{C}$  vanishes, a comparison between (29) and (23) leads to the identification  $\delta_X \mathcal{V}(L, \Xi) = \mathcal{U}(L', X)$ , which cannot be true. We can infer this without any calculation. Indeed, it is well known that if we consider a Lagrangian  $\bar{L} = L + \text{Div } \alpha$  which differs from a given Lagrangian  $L$  only for the addition of a divergence, then the reduced current remains the same, i.e.  $\tilde{\mathcal{E}}(\bar{L}, \Xi) = \tilde{\mathcal{E}}(L, \Xi)$  while the Noether superpotential transforms as follows (see [40, 48]):

$$\mathcal{V}(\bar{L}, \Xi) = \mathcal{V}(L, \Xi) + i_\xi \alpha \quad (30)$$

On the other hand, being equation (23) built out directly from the equations of motion (8), is not affected by the addition of divergence terms to the Lagrangian. For this reason we see that the divergence term  $\text{Div } \mathcal{C}(L, \Xi, X)$  cannot be vanishing in general and it must be sensitive to the representative we choose inside the cohomology class  $[L]$  a given Lagrangian belongs to, in order to counterbalance the transformation rule (30) into (29).

Roughly speaking, we could say that the formalism developed up to now, namely the construction of  $\text{Div } \mathcal{U}(L', X)$  in (23), is nothing but a recipe to algorithmically define the additional term  $\text{Div } \mathcal{C}(L, \Xi, X)$  which has to be added to the Noether superpotential in order to make the variation of conserved quantities insensitive to the choice of a representative inside  $[L]$ .

When dealing with applications we shall see that the fibered morphisms  $\mathcal{U}(L', X)$  and  $\delta_X \mathcal{V}(L, \Xi)$  differ indeed for a term which is nothing but the *covariant* Regge–Teitelboim boundary correction term; see [28, 36, 40, 50, 63].

■

## 4 Purely metric formulation of gravity

Let us now consider the Hilbert Lagrangian density

$$\mathbb{L}_H = \frac{1}{2k} \sqrt{g} g^{\mu\nu} R_{\mu\nu} (j^2 g) \quad (31)$$

( $k = 8\pi G/c^4$ ) or any other Lagrangian differing from it by the addition of boundary terms (e.g. the first order Einstein Lagrangian [22], the first order

covariant Lagrangian [29, 31, 36, 51] or the York's Trace-K Lagrangian [16, 66]). Indeed, what is really relevant is only the expression of Einstein's field equations:

$$-\frac{\sqrt{g}}{2k} G^{\mu\nu} = -\frac{\sqrt{g}}{2k} \left\{ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right\} = 0 \quad (32)$$

from which we define the variational Lagrangian (8):

$$L'(\xi) = \frac{\sqrt{g}}{2k} G^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} ds = \frac{\sqrt{g}}{k} G^{\mu\nu} \nabla_\mu \xi_\nu ds \quad (33)$$

where  $\xi = \xi^\mu \partial_\mu$  denotes a vector field in spacetime.

**First step.** Let us consider a vertical vector field  $X = X_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}} = \delta g_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}}$  on the configuration bundle  $Y = \text{Lor}(M)$  of Lorentzian metrics and let us perform the variation  $\delta_X L'(\xi)$ . By taking into account the relations:

$$\delta R_{\mu\nu} = \nabla_\rho (\delta u_{\mu\nu}^\rho), \quad u_{\mu\nu}^\rho = \gamma_{\mu\nu}^\rho - \delta_{(\mu}^\rho \gamma_{\nu)}^\sigma \quad (34)$$

$$\delta \gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (-\nabla_\sigma \delta g_{\mu\nu} + \nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\sigma\mu}) \quad (35)$$

(where  $\gamma_{\mu\nu}^\rho$  denotes the Levi-Civita connection of the metric  $g$ ) we obtain:

$$\delta_X L'(\xi) = d_\gamma \mathcal{F}^\gamma(j^2 g, j^1 X, j^2 \xi) ds \quad (36)$$

where:

$$\mathcal{F}^\gamma = \mathcal{F}_\rho^\gamma(j^2 g, X) \xi^\rho + \mathcal{F}_\rho^{\gamma\lambda}(j^1 g, j^1 X) \nabla_\lambda \xi^\rho + \mathcal{F}_\rho^{\gamma(\lambda\sigma)}(g, X) \nabla_{(\lambda\sigma)} \xi^\rho \quad (37)$$

with:

$$\begin{aligned} \mathcal{F}_\rho^\gamma &= \frac{\sqrt{g}}{2k} G^{\alpha\beta} \delta g_{\alpha\beta} \delta_\rho^\gamma + \frac{1}{2} T_\sigma^{\gamma[\mu\lambda]} R_{\rho\mu\lambda}^\sigma \\ \mathcal{F}_\rho^{\gamma\lambda} &= \frac{\sqrt{g}}{2k} \delta u_{\mu\nu}^\gamma (2g^{\lambda\mu} \delta_\rho^\nu - g^{\mu\nu} \delta_\rho^\lambda) \\ \mathcal{F}_\rho^{\gamma(\lambda\sigma)} &= T_\rho^{\gamma(\lambda\sigma)} \\ T_\sigma^{\gamma\rho\lambda} &= \frac{\sqrt{g}}{k} \delta g_{\alpha\beta} \left( g^{\beta[\lambda} g^{\rho]\gamma} \delta_\sigma^\alpha + g^{\rho\beta} g^{\alpha[\gamma} \delta_\sigma^{\lambda]} \right. \\ &\quad \left. + \frac{1}{2} g^{\alpha\beta} (g^{\rho[\lambda} \delta_\sigma^{\gamma]} + g^{\gamma[\lambda} \delta_\sigma^{\rho]}) \right) \end{aligned} \quad (38)$$

$$\quad (39)$$

**Second step.** Integrating the expression (37) by parts according to the formulae (18) and (19) we obtain

$$\mathcal{F}^\gamma = \delta_X \tilde{\mathcal{E}}^\gamma(L, \xi) + d_\rho \mathcal{U}^{[\gamma\rho]} \quad (40)$$

where:

$$\tilde{\mathcal{E}}^\gamma(L, \xi) = \frac{\sqrt{g}}{k} G_\nu^\gamma \xi^\nu \quad (41)$$

$$\mathcal{U}^{[\gamma\rho]} = \delta \left\{ \frac{\sqrt{g}}{k} \nabla^{[\rho} \xi^{\gamma]} \right\} + \frac{\sqrt{g}}{k} g^{\mu\nu} \delta u_{\mu\nu}^{[\rho} \xi^{\gamma]} \quad (42)$$

Notice that the reduced current  $\tilde{\mathcal{E}}(L, \xi)$  is clearly vanishing on-shell. Moreover the first term in the right hand side of (42) is nothing but the variation of the Noether superpotential (i.e., in our case, the variation of the Komar superpotential; [57]). On the contrary the second term in the right hand side of (42) is *the covariant Regge-Teitelboim correction term* [10, 36, 40], namely it is the term which has to be added to the Hamiltonian to define a well-posed variational principle. To understand this latter statement, let us analyse (40) in details. Let us choose a (local) section  $g : x \mapsto g_{\mu\nu}(x)$  of the configuration bundle and let us make use of it to pull-back formula (40) onto spacetime  $M$ . Let  $D$  be a (portion of a) Cauchy surface in  $M$  and let the vector field  $\xi$  be transverse to  $D$ . According with our previous definition we identify the integral (of the pull-back) of (40) with the variation of the Hamiltonian:

$$\begin{aligned} \delta_X \mathcal{H}_D(\xi, g) &= \int_D \delta \left\{ \frac{\sqrt{g}}{k} G^\gamma_\nu \xi^\nu \right\} ds_\gamma \\ &+ \int_{\partial D} \left\{ \delta \left[ \frac{\sqrt{g}}{2k} \nabla^{[\rho} \xi^{\gamma]} \right] + \frac{\sqrt{g}}{2k} g^{\mu\nu} \delta u^{[\rho}_{\mu\nu} \xi^{\gamma]} \right\} ds_{\rho\gamma} \end{aligned} \quad (43)$$

Since the vector field  $\xi$  is transverse to  $D$  we can drag (at least locally) the surface  $D$  along the flow of  $\xi$ , defining in this way a  $m$ -dimensional region of spacetime which is foliated into hypersurfaces diffeomorphic to  $D$ . By making use of the (3+1) ADM formalism we can rewrite (43) in terms of quantities which are adapted to the foliation (the details of such a calculation can be found in [10, 15, 40]). We obtain:

$$\delta_X \mathcal{H}_D(\xi, g) = \int_D \left\{ \delta N \mathcal{H} + \delta N^\alpha \mathcal{H}_\alpha + [h_{\alpha\beta}] \delta P^{\alpha\beta} - [P^{\alpha\beta}] \delta h_{\alpha\beta} \right\} d^3x \quad (44)$$

where  $N$  and  $N^\alpha$  are, respectively, the lapse and the shift of the vector field  $\xi = \partial_t$ ,  $\mathcal{H}$  and  $\mathcal{H}_\alpha$  are the usual Hamiltonian constraints of General Relativity,  $P^{\alpha\beta}$  is the momentum conjugated to the metric  $h_{\alpha\beta}$  on the hypersurface  $D$  and  $[h_{\alpha\beta}]$  and  $[P^{\alpha\beta}]$  denote the right hand side of the Hamilton equations:  $\mathcal{L}_\xi h_{\alpha\beta} = [h_{\alpha\beta}]$  and  $\mathcal{L}_\xi P^{\alpha\beta} = [P^{\alpha\beta}]$  (see [10, 60]). We point out that the boundary terms arising in the variation of the first term in the right hand side of (43) exactly cancel out the second and the third boundary terms. Hence we end up with the “pure” bulk term (44) which is the correct expression one would expect for the variation of the Hamiltonian for General Relativity; see [10, 11, 15, 40, 63]. This fact justifies, a posteriori, the definition of variation of the Hamiltonian we have previously attributed to formula (43).

We just outline that the *variation* of the energy  $\delta_X E_D(\xi, g)$  enclosed in the quasilocal region bounded by  $\partial D$  and relative to the pair  $(\xi, g)$  can be defined as the on-shell value of (43), namely:

$$\delta_X E_D(\xi, g) = \int_{\partial D} \left\{ \delta \left[ \frac{\sqrt{g}}{2k} \nabla^{[\rho} \xi^{\gamma]} \right] + \frac{\sqrt{g}}{2k} g^{\mu\nu} \delta u^{[\rho}_{\mu\nu} \xi^{\gamma]} \right\} ds_{\rho\gamma} \quad (45)$$

where, we recall, for *variation* we mean the infinitesimal change of energy when we move along nearby solutions (different vectors  $X$  corresponding to different



paths in the space of solutions all passing through the same “point”  $g(x)$ ). The (3+1) splitting of formula (45) can be found in [40] and for this reason we do not exhibit it here. In [4, 9, 19, 31, 40, 50] the relations interplaying between the integrability conditions of (45) and boundary conditions on the metric and its derivatives are analysed in detail. We just point out that expression (45) with Dirichlet boundary conditions gives rise to the Brown–York quasilocal energy [31, 40]. Moreover the implications of (45) in relation with the first law of black hole thermodynamics have been discussed in [2, 3, 16, 20, 28, 30, 40, 48]. It was there also emphasized how this formula sheds some light on the fundamental contribution of geometry in characterizing the entropy of black hole solutions and even of more general solutions of Einstein’s equations.

For all these reasons we consider expression (45) as an acceptable and viable definition for the (variation of) energy.

## 5 Gravity in tetrad formalism

Let us now consider the formulation of gravity in the tetrad formalism. Even if the tetrad formulation is in many cases equivalent to the purely metric formalism, it becomes necessary when dealing with spinor matter, so that it deserves an investigation in its own.

It is well known that a good mathematical arena to globally describe tetrad gravity is the gauge natural bundle framework (see [27, 41, 59]). In this context the structure bundle of the theory is a principal bundle  $(P, M, p; SO(1, 3))$  over spacetime  $M$ .<sup>5</sup> According to [27] a tetrad field is defined to be a section of a  $GL(4, \mathbb{R})$  bundle  $\Sigma$  which is the bundle associated to the bundle  $P \times L(M)$ , where  $L(M)$  denotes the frame bundle, via the left action

$$\begin{aligned} \rho &: (SO(1, 3) \times GL(4, \mathbb{R})) \times GL(4, \mathbb{R}) \longrightarrow GL(4, \mathbb{R}) \\ \rho &: (\Lambda, J; X) \mapsto \Lambda \cdot X \cdot (J^{-1}) \end{aligned} \quad (46)$$

Fibered coordinates on  $\Sigma$  are denoted by  $(x^\mu, e_\mu^i)$ , where  $i = 0, \dots, 3$ . A tetrad field is then a map  $x \mapsto (x^\mu, e_\mu^i = \theta_\mu^i(x))$ . Given a tetrad field we can define a metric over spacetime through the rule  $g_{\mu\nu} = \eta_{ij} \theta_\mu^i \theta_\nu^j$ ,  $\eta = \text{diag}(-1, 1, 1, 1)$ , so that, a posteriori, the structure bundle  $(P, M, p; SO(1, 3))$  can be identified with the bundle of orthonormal frames  $SO(M, g)$  which is the subbundle of the frame bundle  $L(M)$  formed by all  $g$ -orthonormal frames. We stress that, in this framework, the index  $i$  of the tetrad is not merely a label denoting a set of four 1-forms in spacetime. In this case tetrad fields would be merely a *local* basis of the tangent space (globality being recovered only for parallelizable manifolds) and the theory would just be a natural theory. Global frames, i.e. tetrads, are instead truly gauge natural objects: they are sensitive to the transformations of the structure bundle  $P$ . Each automorphism of the principal bundle functorially induces, via the left action  $\rho$ , a transformation law acting on the tetrad.

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<sup>5</sup>Since we shall not consider here the explicit coupling between gravity and spinor matter the structure group is assumed to be the pseudo orthogonal group instead of its twofold covering group  $Spin(1, 3)$

This transformation law allows to canonically define the Lie derivative of the tetrad fields with respect to any vector field on the structure bundle. Indeed a projectable vector field  $\Xi_P$  in the principal bundle, locally described as

$$\Xi_P = \xi^\mu(x)\partial_\mu + \xi^{ij}(x)\rho_{ij}, \quad \xi^{ij} = \xi^{[ij]} \quad (47)$$

(having denoted by  $\rho = g\partial/\partial g$ ,  $g \in SO(1,3)$  a basis for right invariant vector fields on  $P$  in a trivialization  $(x, g)$  of  $P$ ) canonically induces the projectable vector field  $\Xi$  on  $\Sigma$  given by  $\Xi = \xi^\mu\partial_\mu + \xi_j^i e_\mu^j \partial/\partial e_\mu^i$ . Through the vector field  $\Xi$  we can define the formal Lie derivative  $\mathcal{L}_\Xi e : J^1\Sigma \longrightarrow V(\Sigma)$ . It is intrinsically defined by the rule  $\mathcal{L}_\Xi e \circ j^1\theta := \mathcal{L}_\Xi\theta = T\theta \circ \xi - \Xi \circ \theta$ , where  $\xi$  denotes the projection of  $\Xi$  onto  $M$  and  $\theta$  is a tetrad field. In coordinates:

$$\mathcal{L}_\Xi e_\mu^i = \xi^\nu d_\nu e_\mu^i + d_\mu \xi^\nu e_\nu^i - \xi_j^i e_\mu^j \quad (48)$$

In the tetrad affine formulation of gravity the configuration bundle  $Y$  is assumed to be the bundle:  $Y = \Sigma \longrightarrow M$ . The Lagrangian of the theory turns out to be the fibered morphism  $L : J^2\Sigma \longrightarrow \Lambda^4(M)$  locally described through the Lagrangian density:

$$\mathbb{L} = \frac{1}{8k} e_\mu^i e_\nu^j R_{\alpha\beta}^{kl} (j^2 e) \epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} \quad (49)$$

where  $R^{kl}(j^2 e) = d\omega^{kl} + \omega^k_h \wedge \omega^{hl}$  is the field strength of the Levi-Civita spin connection  $\omega^{kl}$ . The theory is invariant under the whole group  $\text{Aut}(P)$  meaning that each vector field (47) is an infinitesimal generator of symmetries. Field equations ensuing from (49) are,

$$\frac{1}{4k} e_\nu^j R_{\alpha\beta}^{kl} \epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} = 0 \quad (50)$$

As it is well known the theory described by (50) is completely equivalent to Einstein's theory (provided that  $\det(e_\mu^i) \neq 0$ ), the difference between the two sets of field equations being only a matter of notation. Therefore, we should expect to reproduce the same expressions for the conserved quantities we derived in the previous section.

The variational Lagrangian associated with the Euler-Lagrange equations (50) and the infinitesimal generator of symmetries (47) turns out to be:

$$L'(\Xi) = -\frac{1}{4k} \epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} \{ e_\nu^j R_{\alpha\beta}^{kl} \mathcal{L}_\Xi e_\mu^i \} ds \quad (51)$$

where the Lie derivative of the tetrad field has been defined by (48). It is an easy calculation to show that (51) coincides with (33) so that exactly the same results of the previous section are recovered. Notice that, even though this is exactly the result we physically expected, it is nevertheless a result not so obvious from a mathematical viewpoint. Indeed, even if the Lagrangians (31) and (49) are equivalent as far as their field equations contents are concerned, they describe instead different theories from a geometrical viewpoint, since the former theory

is a natural theory while the latter one is a true gauge natural theory. Therefore they admit two different groups of symmetries given, respectively, by  $\text{Diff}(M)$  and  $\text{Aut}(P)$ . For this reason conserved quantities for tetrad gravity calculated via Noether theorem are *intrinsically indeterminate* from a mathematical viewpoint (see [59]) since the components  $\xi^{ij}$  of the vector field (47) enter into the Noether superpotential and they can be, a priori, fixed arbitrarily. This indeterminacy does not occur in our framework since, in (51), the skew-symmetric components  $\xi^{ij}$  are contracted with the symmetric components  $R_{\mu\nu}e_i^\mu e_j^\nu$  of the Ricci tensor and therefore they give rise to an identically vanishing term.

Therefore, we expect that in any generalization of Einstein's theory involving also the non-symmetric part of the Ricci tensor (see, e.g. [37, 45]) the indeterminacy will still remain so that we have to somehow bypass it. This is the issue we shall consider in the next section.

## 6 Einstein Cartan theory and the Kosmann lift

The Einstein–Cartan (Sciama–Kibble) theory, ECKS from now on, describes a generalization of Einstein theory of gravitation in which the spin angular momentum of matter plays a dynamical role. In the ECKS model the dynamical connection is still metric compatible but it is no longer symmetric, the torsion part being coupled to the spin of matter. Therefore the theory leads to deviations from General Relativity (even though the mathematical differences can be physically appreciated only in extreme solutions); see [45].

The ECKS theory can be framed in the domain of gauge natural theories; see [59]. The configuration bundle  $Y$  for the gravitational ECKS Lagrangian is assumed to be the product bundle:

$$Y = \Sigma \times \mathcal{C}(P) \longrightarrow M \quad (52)$$

where  $\mathcal{C}(P) = J^1P/SO(1,3)$  denotes the bundle of principal connections over the structure bundle  $P$  (see [56]) while  $\Sigma$  and  $P$  have been defined in the previous section. The bundle  $Y$  is clearly a gauge natural bundle associated with the structure bundle  $(P, M, p; SO(1,3))$ . Fibered coordinates on  $Y$  are denoted by  $(x^\rho, e_\mu^i, A_\nu^{jk})$ . The Lagrangian of the theory is the sum  $L = L_{EC} + L_M$  of the gravitational ECKS Lagrangian  $L_{EC}$  and the matter Lagrangian  $L_M$ . The former Lagrangian turns out to be the fibered morphism  $L_{EC} : \Sigma \times J^1\mathcal{C}(P) \longrightarrow \Lambda^4(M)$  locally described through the Lagrangian density:

$$L_{EC} = \frac{1}{4} e_\mu^i e_\nu^j F_{\alpha\beta}^{kl} (j^1 A) \epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} \quad (53)$$

where  $F^{kl}(j^1 A) = dA^{kl} + A^k_h \wedge A^{hl}$  is the field strength of the connection (from now on we shall omit the constant factor  $1/2k$  in front of the Lagrangian). The theory is invariant under the whole group  $\text{Aut}(P)$  (provided that the matter Lagrangian is, in its turn, a gauge natural Lagrangian) meaning that each vector field (47) is an infinitesimal generator of symmetries. Field equations ensuing

from (53) through the variations of the independent fields  $e_\mu^i$  and  $A_\nu^{jk}$  are, respectively:

$$\frac{1}{2}e_\nu^j F_{\alpha\beta}^{kl} \epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} = \tau_i^\mu \quad (54)$$

$$-\epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} e_\mu^i \overset{A}{D}_\alpha e_\nu^j = \tau_{kl}^\beta \quad (55)$$

where  $\overset{A}{D}_\alpha e_\mu^i = d_\alpha e_\mu^i + A^i_{j\alpha} e_\mu^j$ , while  $\tau_i^\mu$  and  $\tau_{kl}^\beta$  are, respectively, the energy momentum and the spin of matter, the explicit form of which depends on the matter Lagrangian.

As it is well known, if the spin of matter is zero the theory described by (54) and (55) is on-shell equivalent to Einstein's theory (provided that  $\det(e_\mu^i) \neq 0$ ). Indeed equation (55) admits the solution  $A^i_{j\alpha} = \omega^i_{j\alpha}(j^1 e)$ , where  $\omega^i_{j\alpha}(j^1 e)$  denotes the Levi-Civita connection built out of the tetrad together with its derivatives. This solution inserted back into the Lagrangian (53) gives rise to the Hilbert Lagrangian (31) while equations (54) become Einstein's equations (32). Nevertheless, even when  $\tau_{kl}^\beta = 0$  the equivalence holds true only on-shell since there does not exist any off-shell relation between the tetrad field and the connection. On the contrary, when the spin of matter does not vanish equation (55) becomes an algebraic equation relating the “non-metricity” of the connection (namely, the torsion) with the spin matter content.

We shall now examine the energetic informations ensuing from the pure gravitational part of the theory (left hand side of equations (54) and (55)). Since all calculations will be performed off-shell, the result will be not affected by the specific distribution of spinning matter one is taking into consideration. The variational Lagrangian associated with the left hand side of Euler-Lagrange equations (54)–(55) and the infinitesimal generator of symmetries (47) turns out to be:

$$L'(\Xi) = \epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} \left\{ -\frac{1}{2} e_\nu^j F_{\alpha\beta}^{kl} \mathcal{L} \Xi e_\mu^i + e_\mu^i \overset{A}{D}_\alpha e_\nu^j \mathcal{L} \Xi A_\beta^{kl} \right\} ds \quad (56)$$

where the Lie derivative of the tetrad field has been defined in (48), while

$$\begin{aligned} \mathcal{L}_\Xi A_\beta^{kl} &= \xi^\rho d_\rho A_\beta^{kl} + d_\beta \xi^\rho A_\rho^{kl} + \overset{A}{D}_\beta \xi^{kl} \\ &= \xi^\rho F_{\rho\beta}^{kl} + \overset{A}{D}_\beta \xi_{(V)}^{kl} \quad (\xi_{(V)}^{kl} = \xi^{kl} + A_\rho^{kl} \xi^\rho) \end{aligned} \quad (57)$$

Notice that the Lagrangian (56) is a gauge natural Lagrangian by its own; see section 2.

**First step.** The variation of (56) with respect to a vertical vector field  $X = \delta e_\mu^i \partial / \partial e_\mu^i + \delta A_\mu^{ij} \partial / \partial A_\mu^{ij}$  splits into a term  $\langle e(L') | X \rangle$  which is identically vanishing plus a divergence term  $\text{Div } \mathcal{F}(L', X)$  (see (13)). The term  $\mathcal{F}(L', X)$  is linear in the coefficients  $\xi^\mu, \xi^{ij}$  and their derivatives  $d_\mu \xi^\nu, \overset{A}{D}_\mu \xi^{ij}$ . Integrating by parts according to formula (18) and (19) we get (**second step**):

$$\mathcal{F}^\gamma(L', X) = \delta_X \tilde{\mathcal{E}}^\gamma(L, \Xi) + d_\beta \mathcal{U}^{\gamma\beta}(L', X) \quad (58)$$

where:

$$\tilde{\mathcal{E}}^\gamma(L, \Xi) = \epsilon^{\gamma\nu\alpha\beta} \epsilon_{ijkl} \left\{ -\frac{1}{2} e_\nu^j F_{\alpha\beta}^{kl} e_\rho^i \xi^\rho - e_\beta^i \overset{A}{D}_\alpha e_\nu^j \xi_{(V)}^{kl} \right\} \quad (59)$$

$$\mathcal{U}^{\gamma\beta}(L', X) = \epsilon^{\gamma\beta\mu\nu} \epsilon_{ijkl} e_\nu^j \left\{ \delta A^{kl}{}_\mu e_\rho^i \xi^\rho + \delta e_\mu^i \xi_{(V)}^{kl} \right\} \quad (60)$$

Therefore, the variation of conserved quantities is defined (see (24)) as the on-shell integral:

$$\begin{aligned} \delta_X \mathcal{Q}_D(\Xi, e, A) &\simeq \frac{1}{2} \int_{\partial D} \mathcal{U}^{\gamma\beta}(L', X) ds_{\gamma\beta} \\ &\simeq \frac{1}{2} \int_{\partial D} \epsilon^{\gamma\beta\mu\nu} \epsilon_{ijkl} e_\nu^j \left\{ \delta A^{kl}{}_\mu e_\rho^i \xi^\rho + \delta e_\mu^i \xi_{(V)}^{kl} \right\} ds_{\gamma\beta} \end{aligned} \quad (61)$$

where  $\partial D$  is the two dimensional boundary of a three dimensional region  $D$ . Notice that the components  $\xi^{ij}$  of the vector field enter into the definition of conserved quantities. Since, up to now, no preferred condition can be mathematically imposed on those components, formula (61) features an intrinsic indeterminacy. Nevertheless it was suggested in [59] that the indeterminacy can be eliminated if the vertical components  $\xi_{(V)}^{kl}$  are defined through the Kosmann lift (we shall enter into details later on). The choice of the Kosmann lift was justified, a posteriori, in [59] from a physical viewpoint, since, in an apparently surprising way, it is the only lift among the possible ones which allows one to exactly reproduce the expected values for conserved quantities in explicit applications; see [3, 29]. Our next issue will be then to give a mathematical justification for this choice. Indeed we shall show that the Kosmann lift arises spontaneously when trying to restore the naturality of the gauge natural variational Lagrangian (56).

To this end let us face up to the problem of calculating, through the variational Lagrangian (56), the variation of the conserved quantity associated to a spacetime vector field  $\xi$ . Notice, however, that vertical vector fields  $\Xi_P = \xi^{ij}(x)\rho_{ij}$  in the structure bundle (see (47)) are well defined objects and they describe infinitesimal gauge transformations (and, accordingly, expression (61) describes the variation of gauge charges when calculated for vertical vector fields). On the contrary, horizontal vector fields  $\Xi_P = \xi^\mu(x)\partial_\mu$  are not globally well defined since they do not transform tensorially. Roughly speaking this means that the Lie derivatives of the dynamical fields are defined only with respect to vector fields in  $P$  and *not* with respect to spacetime vector fields. Hence, we have, first of all, to somehow lift the vector field  $\xi$  to a vector field  $\Xi = \hat{\xi}$  on the principal bundle  $P$ . If we are able to do this, the Lie derivatives  $\mathcal{L}_{\hat{\xi}}A$  and  $\mathcal{L}_{\hat{\xi}}e$  in expression (56) are well defined and expression (61), with  $\Xi = \hat{\xi}$ , can be used to define global charges (such as energy, momentum and angular momentum). What we shall look for is, if it exists, a geometrically well defined lift which restores, off-shell, the naturality of the variational Lagrangian (56).

Notice that, as far as expression (56) is concerned there does not appear any preferred dynamical linear connection on spacetime (as it stands, expression

(56) already transforms covariantly with respect to the automorphisms of the principal bundle). Hence we are free to introduce, by hands and according to our convenience, a linear connection  $\Gamma$  built out of the dynamical fields. In this way we can extend the  $SO(1,3)$  covariant derivative  $\overset{A}{D}$ , which acts on internal (Latin) indices, to a  $SO(1,3) \times GL(4)$  covariant derivative  $\overset{\Gamma}{\nabla}$  which acts on internal as well as spacetime (Greek) indices. We define the linear connection  $\Gamma$  by requiring that  $\overset{\Gamma}{\nabla}$  annihilates the tetrad field, i.e.:

$$\overset{\Gamma}{\nabla}_\gamma e_\mu^i := \overset{A}{D}_\gamma e_\mu^i + \Gamma^\rho_{\mu\gamma} e_\rho^i = d_\gamma e_\mu^i + \Gamma^\rho_{\mu\gamma} e_\rho^i + A^i_{j\gamma} e_\mu^j = 0 \quad (62)$$

These are 64 independent linear equations in the 64 variables  $\Gamma^\rho_{\mu\gamma}$ . Thereby the coefficients of the linear connection, being uniquely determined in terms of the principal connection and of the tetrad field together with its derivatives, describe in fact a dynamical field. Notice that from (62) it immediately follows that  $g$  is parallel along  $\Gamma$ , i.e.:

$$\overset{\Gamma}{\nabla}_\gamma g_{\mu\nu} = 0 \quad (63)$$

Hence the connection turns out to be:

$$\Gamma^\rho_{\mu\gamma} = \gamma^\rho_{\mu\gamma}(j^1 g) + K^\rho_{\mu\gamma} \quad (64)$$

where  $\gamma^\rho_{\mu\gamma}(j^1 g)$  denotes the Levi-Civita connection of the metric  $g_{\mu\nu} = \eta_{ij} e_\mu^i e_\nu^j$  while  $K^\rho_{\mu\gamma}$  is the so-called *contorsion tensor*:

$$K^\rho_{\mu\gamma} = T^\rho_{\mu\gamma} - T_\mu^\rho{}_\gamma - T_\gamma^\rho{}_\mu, \quad T^\rho_{\mu\gamma} = \Gamma^\rho_{[\mu\gamma]} \quad (65)$$

(indices are clearly raised and lowered with the metric  $g_{\mu\nu}$  and its inverse). In this step we can replace the original 24 variables  $A^i_{j\gamma}$  with the 24 variables  $K^\rho_{\mu\gamma}$  (notice that  $K_{\rho\mu\gamma} = -K_{\mu\rho\gamma}$ ).

Equation (62) allows to produce a relation between the field strength  $F^{ij}_{\mu\nu}$  and the Riemann tensor

$$R^\alpha_{\beta\mu\nu} = d_\mu \Gamma^\alpha_{\beta\nu} - d_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\beta\mu} \quad (66)$$

The relation is indeed established by the property:

$$\begin{aligned} 0 = [\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu] e_\rho^i &= F^i_{j\mu\nu} e_\rho^j - R^\gamma_{\rho\mu\nu} e_\gamma^i + 2T^\gamma_{\mu\nu} \overset{\Gamma}{\nabla}_\gamma e_\rho^i = \\ &= F^i_{j\mu\nu} e_\rho^j - R^\gamma_{\rho\mu\nu} e_\gamma^i \end{aligned} \quad (67)$$

which can be easily solved in terms of the field strength as follows:

$$F^i_{j\mu\nu} = R^\gamma_{\rho\mu\nu} e_\gamma^i e_j^\rho \quad (68)$$

Inserting (62) and (68) into the variational Lagrangian (56) we obtain:

$$L'(\Xi) = \left\{ 2\sqrt{g} (R^\mu{}_\rho - \frac{1}{2}\delta^\mu_\rho R) e_i^\rho \mathcal{L}_\Xi e_\mu^i + \epsilon^{\mu\nu\alpha\beta} \epsilon_{ijkl} e_\mu^i e_\rho^j T^\rho{}_{\nu\alpha} \mathcal{L}_\Xi A_{\beta}^{kl} \right\} ds \quad (69)$$

where  $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$  denotes the Ricci tensor built out of the connection  $\Gamma$ . From (64) it follows that:

$$R_{\mu\nu}(j^1\Gamma) = R_{\mu\nu}(j^1\gamma) + \nabla_\beta^\gamma K^\beta{}_{\mu\nu} - \nabla_\nu^\gamma K^\beta{}_{\mu\beta} + K^\beta{}_{\sigma\beta} K^\sigma{}_{\mu\nu} - K^\alpha{}_{\sigma\nu} K^\sigma{}_{\mu\alpha} \quad (70)$$

where  $R_{\mu\nu}(j^1\gamma)$  is the Ricci tensor of the Levi-Civita connection and  $\nabla^\gamma$  is the (metric) covariant derivative induced by  $\gamma$ . Notice that  $R_{\mu\nu}$  is not symmetric. By splitting the first term in the right hand side of (69) into its symmetric and skew-symmetric parts we obtain:

$$\begin{aligned} L'(\Xi) = & 2\sqrt{g} \left\{ G^{(\mu\rho)} e_{\rho i} \mathcal{L}_\Xi e_\mu^i + \delta_{\rho\gamma\lambda}^{\nu\alpha\beta} e_k^\gamma e_l^\lambda T^\rho{}_{\nu\alpha} \mathcal{L}_\Xi A_\beta^{kl} \right\} ds \\ & + 2\sqrt{g} R^{[\mu\rho]} e_{\rho i} \mathcal{L}_\Xi e_\mu^i ds \end{aligned} \quad (71)$$

with  $G^{\mu\rho} = R^{\mu\rho} - 1/2 g^{\mu\rho} R$ . The above expression suggests how to select a preferred lift  $\Xi$  of the spacetime vector field  $\xi$ , namely, how to define the components  $\xi_j^i$  in (48) in terms of the components  $\xi^\mu$ . We simply require that the latter term in (71) does vanish, i.e:

$$e_{i[\rho} \mathcal{L}_\Xi e_{\mu]}^i = 0 \quad (72)$$

This algebraic equation can be easily and uniquely solved in terms of the components  $\xi_j^i$  yielding:

$$\xi^{ij} = \xi^{[ij]} = e_\alpha^{[i} e^{j]\lambda} d_\lambda \xi^\alpha - \xi^\gamma e_\mu^{[i} d_\gamma e^{j]\mu} \quad (73)$$

The above expression is known as *the generalized Kosmann lift*. We remark that the Kosmann lift was defined for the first time in [27] in order to establish a relationship between the ad hoc definition of Lie derivative of spinor fields given in [58] and the general theory of Lie derivatives on fiber bundles. We stress that, in our framework, the generalized Kosmann lift arises, in a completely independent way, when trying to restore naturality in the variational Lagrangian (56). Namely, it is the consequence of the algebraic equation (72) and it is not an *ad hoc* definition. We also stress that the solution (73) is globally well defined. Indeed, the Kosmann lift  $K(\xi)$  of a spacetime vector field  $\xi$ , which in a system of fibered coordinates read as follows:

$$K(\xi) = \xi^\mu \partial_\mu + \left\{ e_\alpha^{[i} e^{j]\lambda} d_\lambda \xi^\alpha - \xi^\gamma e_\mu^{[i} d_\gamma e^{j]\mu} \right\} \rho_{ij} \quad (74)$$

transforms tensorially (as can be inferred through a direct inspection) and can thereby globally defined. Namely, all the local expressions (74) can be patched together to define a global vector field.<sup>6</sup>

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<sup>6</sup>The geometric nature of the Kosmann lift can be easily understood as follows. One starts with the spacetime vector field  $\xi$  and considers its natural lift  $L(\xi)$  to the frame bundle  $L(M)$ . Fixing a frame  $e = (e_\mu^i)$  and thereby a spacetime metric  $g$  one can construct the bundle of orthonormal frames  $SO(M, g)$  which is a subbundle of  $L(M)$ . The Kosmann lift is nothing but the projection of  $L(\xi)$  from the tangent space of  $L(M)$  onto the tangent space of  $SO(M, g)$ ; see [41].

We now analyze the consequences that the expression (73) has on our theory. Inserting (73) into (48) we have:

$$\mathcal{L}_{K(\xi)} e_\mu^i = \frac{1}{2} e^{i\nu} \mathcal{L}_\xi g_{\mu\nu} \quad (75)$$

Instead if we insert (73) into (57) and we take the relations (62) and (68) into account we get:

$$\begin{aligned} \mathcal{L}_{K(\xi)} A_\mu^{jk} &= e_\lambda^{[j} e^{k]\gamma} \left( \xi^\sigma R^\lambda_{\gamma\sigma\mu} + \tilde{\nabla}_\mu^\Gamma \tilde{\nabla}_\gamma \xi^\lambda \right) \\ &= e_\lambda^{[j} e^{k]\gamma} \mathcal{L}_\xi \Gamma^\lambda_{\gamma\mu} \end{aligned} \quad (76)$$

where  $\tilde{\nabla}$  denotes the covariant derivatives with respect to the trasposed connection  $\tilde{\Gamma}^\lambda_{\gamma\mu} = \Gamma^\lambda_{\mu\gamma}$ . Finally, if we insert (72), (75) and (76) into (71) we end up with:

$$L'(K(\xi)) = \sqrt{g} \{ G^{\mu\rho} \mathcal{L}_\xi g_{\mu\rho} + 2g^{\sigma\gamma} \mathcal{T}^\beta_{\lambda\sigma} \mathcal{L}_\xi \Gamma^\lambda_{\gamma\beta} \} ds \quad (77)$$

where:

$$\mathcal{T}^\beta_{\lambda\sigma} = T^\beta_{\lambda\sigma} - \delta_\sigma^\beta T^\rho_{\lambda\rho} + \delta_\lambda^\beta T^\rho_{\sigma\rho} \quad (78)$$

is the *modified* torsion tensor; see [45]. Notice that the variational Lagrangian of our initial theory turns out to be completely natural. Notice also that (77) is nothing but the variational Lagrangian ensuing from the Einstein–Cartan Lagrangian density in the metric affine formalism:

$$\mathbf{L}_{EC} = \sqrt{g} g^{\mu\nu} R_{\mu\nu}(j^1\Gamma) \quad (79)$$

where  $\Gamma$  is the metric compatible connection (64). Indeed field equations ensuing from (79) through a variational principle *à la* Palatini are nothing but the coefficients in front of the Lie derivatives in (77) (except for a sign), i.e.

$$\frac{\delta \mathbf{L}_{EC}}{\delta g_{\mu\rho}} = -\sqrt{g} \left\{ R^{(\mu\rho)} - \frac{1}{2} g^{\mu\rho} R \right\} \quad (80)$$

$$\frac{\delta \mathbf{L}_{EC}}{\delta \Gamma^\lambda_{\gamma\beta}} = -2\sqrt{g} g^{\sigma\gamma} \mathcal{T}^\beta_{\lambda\sigma} \quad (81)$$

**Remark 6.1** We point out that only the symmetric components of the Ricci tensor enter into the Lagrangian (79) and in its associated variational Lagrangian (77). This fact justifies a posteriori the condition (72). ■

Notice, however, that in order to extract the information about the variation of conserved quantities from the variational Lagrangian (77) we have to take into account that the linear connection  $\Gamma$  is a dynamical field  $\Gamma(j^1g, K)$  depending on the metric, its derivatives and the contorsion tensor; see (64).



In other words (when implementing the first step) the variational derivatives of the Lagrangian (77) are to be performed with respect to the independent fields  $g_{\mu\nu}$  and  $K^\alpha_{\beta\mu}$  (or, equivalently, with respect to  $g_{\mu\nu}$  and  $\mathcal{T}^\alpha_{\beta\mu}$ ). After a straightforward calculation we get:

$$\sqrt{g} G^{\mu\rho} \mathcal{L}_\xi g_{\mu\rho} + 2g^{\sigma\gamma} \mathcal{T}^\beta_{\lambda\sigma} \mathcal{L}_\xi \Gamma^\lambda_{\gamma\beta} = L_1 + L_2 + L_3 \quad (82)$$

where:

$$L_1 = \sqrt{g} G^{\mu\nu} (j^2 g) \mathcal{L}_\xi g_{\mu\nu} \quad (83)$$

$$L_2 = \mathcal{L}_\xi \{ -\sqrt{g} g^{\mu\nu} (K^\rho_{\sigma\rho} K^\sigma_{\mu\nu} - K^\alpha_{\sigma\nu} K^\sigma_{\mu\alpha}) \} \quad (84)$$

$$L_3 = d_\beta \{ -2\sqrt{g} \mathcal{T}^{\mu\beta\nu} \mathcal{L}_\xi g_{\mu\nu} \} \quad (85)$$

The potential for  $L_1$  has already been computed in section 4 and is given (apart from a factor  $2k$ ) by formula (42). Moreover there is no contribution to the potential from  $L_2$  since  $L_2$  affects only the term  $\delta\tilde{\mathcal{E}}$  in (23). Indeed we have (**first step**):

$$\delta L_2 = d_\rho \{ -\xi^\rho \delta (\sqrt{g} g^{\mu\nu} (K^\alpha_{\sigma\alpha} K^\sigma_{\mu\nu} - K^\alpha_{\sigma\nu} K^\sigma_{\mu\alpha})) \} \quad (86)$$

hence, see (13):

$$\mathcal{F}^\rho(L_2, \delta g, \delta K) = -\xi^\rho \delta (\sqrt{g} g^{\mu\nu} (K^\alpha_{\sigma\alpha} K^\sigma_{\mu\nu} - K^\alpha_{\sigma\nu} K^\sigma_{\mu\alpha})) \quad (87)$$

The latter expression is linear in  $\xi$  so that, in the second step, no further integration with respect to  $\xi$  can be made, namely no divergence term entering into the total potential comes out from it.

The potential receives instead a contribution from the term  $L_3$ . To obtain it we have to consider the part enclosed between the braces in (85), develop explicitly the Lie derivatives and then apply (19) with  $h+r-1=1$ . The result turns out to be

$$\mathcal{U}^{\beta\alpha}(L_3, \delta g, \delta \mathcal{T}) = \delta \left\{ 2\sqrt{g} (\mathcal{T}^{[\alpha}_{\rho}{}^{\beta]} + \mathcal{T}^{\alpha\beta}_{\rho}) \xi^\rho \right\} \quad (88)$$

Hence the total potential for the variational Lagrangian (77) is given by the sum of the purely metric potential (42) plus (88), namely:

$$\begin{aligned} \mathcal{U}^{[\beta\alpha]} &= \delta \left\{ 2\sqrt{g} \nabla^{[\alpha} \xi^{\beta]} \right\} + 2\sqrt{g} g^{\mu\nu} \delta u^{[\alpha}_{\mu\nu} \xi^{\beta]} \\ &+ \delta \left\{ 2\sqrt{g} (\mathcal{T}^{[\alpha}_{\rho}{}^{\beta]} + \mathcal{T}^{\alpha\beta}_{\rho}) \xi^\rho \right\} \end{aligned} \quad (89)$$

where  $u^\rho_{\mu\nu} = \gamma^\rho_{\mu\nu} - \delta^\rho_{(\mu} \gamma^\sigma_{\nu)\sigma}$ .<sup>7</sup> We remark that, if the spin contribution of the matter is zero, it holds true that  $\mathcal{T}^\rho_{\lambda\sigma} \simeq 0$ . Hence, expression (89) reproduces on-shell exactly expression (42). When the spin contribution of matter is not vanishing it gives to conserved quantities a geometric contribution (i.e. relative

<sup>7</sup>The same result (89) could be also achieved, in a quite more involved way, by inserting the expression (73) of the Kosmann lift directly into (60)

to the pure gravitational part of the theory) which is produced by the torsion terms in (89). Notice however that the equation (55) is an algebraic relation between torsion and spin matter meaning that, outside the distribution of matter, there is no torsion, i.e. torsion does not propagate (see [45]). The only way the spin of matter affects conserved quantities outside the distribution of matter (where (89) collapses into (42)) is through its influence on the metric tensor.

## 7 Conclusions and perspectives

We have described a theory which algorithmically defines the variation of conserved quantities in the realm of gauge natural theories. The formalism is developed through a two steps procedure starting from the variational Lagrangian and by making use of techniques related to the Calculus of Variations in the jet bundle framework. We have tested the viability of the results the theory predicts by applying them to gravitational theories. In the classical Einstein's metric formulation we reproduced results already found elsewhere. Moreover, when dealing with ECKS theory, viewed as a gauge natural generalization of gravity, a preferred lift of spacetime vector fields, namely the Kosmann lift, is automatically selected by the formalism itself in order to restore naturality in the gauge natural framework. We clearly welcome this additional unlooked-for result which has turned up from the theory since it seems to endow the Kosmann lift, that in literature was previously justified only on physical grounds (see [3, 29, 59]), also with a good mathematical reason of existence.

Moreover we point out that the left hand side of equations (54) and (55) can be easily generalized to encompass any spacetime dimension  $D$ ,  $D > 2$ . In particular, their generalization to odd dimensions reproduces the field equations content of Chern-Simon theories of the group  $SO(2, D-1)$  or  $SO(1, D)$ , see [18]. The case  $D = 3$  has been analyzed in [3] following the spirit of the formalism presented here. We believe that it will be worth investigating also the case  $D = 5$  (as well as any larger dimension), mainly because of its relationship with the Gauss-Bonnet generalization of Einstein's gravity, a generalization which seems to be relevant in effective theories ensuing from the low-energy limit of string theory (see [43]).

We also stress that our recipe is well suited to be generalized to a more general class of symmetries other than those induced just by projectable vector fields on the structure bundle. Indeed the fundamental requirement we made on the Lie derivatives of the fields is that they can be expanded as a linear combination of the independent parameters  $\xi^\mu$  and  $\xi^A$  together with their derivatives up to a fixed (finite) order. This is in fact the assumption which makes the formalism algorithmically well-defined (see expression (9) and (17)). No restriction has been instead imposed on the coefficients of the aforementioned expansion which, in principle, can depend not only on the field together with their first order derivatives, but also on an arbitrary (but finite) number of derivatives. Namely, symmetries with respect to generalized vector fields (see [26, 62] are not out of the scope of the formalism here developed. Thereby we guess that the

formalism is in good position to deal with the so-called generalized symmetries, e.g. supersymmetries and BRST transformations; see [32, 46].

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## References

- [1] A.Achúcarro, P.K.Townsend, Phys.Lett. **B 180** (1986), 89;  
M. Bañados, T. Brotz, M. E. Ortiz, Nucl.Phys. **B545** (1999) 340-370, (hep-th/9802076);  
M. Bañados, Invited talk at the Second Meeting "Trends in Theoretical Physics", held in Buenos Aires, December 1998, (hep-th/9901148); E. Witten, *Nucl. Phys.* **B311** (1988), 46; E. Witten, *Commun. Math. Phys.* **121** (1989), 351.
- [2] G. Allemandi, M. Francaviglia, M. Raiteri, Class. Quantum Grav. **19** (2002), 2633 (gr-qc/0110104).
- [3] G.Allemandi, M.Francaviglia, M. Raiteri, Class. Quantum Grav. **20** (2003) 483 (gr-qc/0211098).
- [4] S.C.Anco, R.S.Tung, J. Math. Phys **43** (2002), 3984;  
S.C.Anco, R.S.Tung, J. Math. Phys **43** (2002), 5531.
- [5] I.M. Anderson, C.G. Torre, Phys. Rev. Lett. **77** (1996) 4109 (hep-th/9608008);  
C.G. Torre, hep-th/9706092, Lectures given at 2nd Mexican School on Gravitation and Mathematical Physics, Tlaxcala, Mexico (1996).
- [6] R. Arnowitt, S. Deser and C. W. Misner, in: *Gravitation: An Introduction to Current Research*, L. Witten ed. Weley, New York, 1962.
- [7] A. Ashtekar, *New Perspectives in Canonical Gravity*, Bibliopolis (Napoli, 1988).
- [8] A. Ashtekar, L. Bombelli, O.Reula, in: *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, Editor: M. Francaviglia, Elsevier Science Publishers B.V., (Amsterdam, 1991) 417.
- [9] G. Barnich, F. Brandt, Nucl.Phys. **B633** (2002) 3 (hep-th/0111246).
- [10] I.Booth, PhD Thesis from University of Waterloo, gr-qc/0008030;  
I.Booth, Class.Quant.Grav. **18** (2001) 4239 (gr-qc/0105009).

- [11] I. Booth, S. Fairhurst, gr-qc/0301123.
- [12] A. Borowiec, M. Ferraris, M. Francaviglia, in press in J. Phys. A (hep-th/0301146).
- [13] J.D.Brown, J.Creighton, R.B.Mann, Phys. Rev. D **50** (10) (1994), 6394.
- [14] J. D. Brown, M. Henneaux, Comm. Math. Phys. **104** (1986), 207.
- [15] J. D. Brown, S.R. Lau, J. W. York, to be published in Annals of Physics (gr-qc/0010024).
- [16] J. D. Brown, J. W. York, Phys. Rev. D **47** (4), (1993),1407;  
J. D. Brown, J. W. York, Phys. Rev. D **47** (4) (1993), 1420.
- [17] G. Burnett, R.M. Wald, Proc. Roy. Soc. Lond., A**430** (1990), 56;  
J. Lee, R.M. Wald, J. Math. Phys **31** (1990), 725;  
R.M. Wald, A. Zoupas, Phys. Rev. D **61** (2000), 084027.
- [18] A. H. Chamseddine and J. Fröhlich, *Commun. Math. Phys.* **147** (1992), 549;  
A. H. Chamseddine, *Nucl. Phys.* **B 346** (1990), 213;  
M. Bañados, C. Teitelboim, J. Zanelli, Phys.Rev. **D49** (1994) 975, (gr-qc/9307033)
- [19] C.-M. Chen, J. M. Nester, Gravitation & Cosmology **6**, (2000), 257 (gr-qc/0001088).
- [20] R. Clarkson, L. Fatibene, R.B. Mann, Nucl.Phys. **B652** (2003) 348 (hep-th/0210280).
- [21] D.J.Eck, Mem. Amer. Math. Soc. **33** (1981), 247.
- [22] A. Einstein, *Sitzungsber. Preuss. Akad. Wiss. (Berlin)* (1916), 1111.
- [23] A. Einstein, *Sitzungsber. Preuss. Akad. Wiss. (Berlin)* (1923), 137.
- [24] A. Einstein, *Sitzungsber. Preuss. Akad. Wiss. (Berlin)* (1925), 414.
- [25] L. Fatibene, M. Ferraris, M. Francaviglia, J. Math. Phys. **38** (1997), 3953;
- [26] L.Fatibene, M. Ferraris, M. Francaviglia, R. McLenaghan, J. Math. Phys. **43** (2002) 3147;  
G.Giachetta, L.Mangiarotti, G.Sardanashvily, math-ph/0304025.
- [27] L. Fatibene, M. Ferraris, M. Francaviglia, M.Godina in: *Proc. 6th International Conference on Differential Geometry and its Applications* , J.Janyška, I.Kolar, J.slovak Eds. (Brno, Czech Republic, 1995) 549–558;  
L. Fatibene, M. Ferraris, M. Francaviglia, M. Godina, Gen. Relativ. Grav. **30** (1998), 1371.

- [28] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *Annals of Phys.* **275** (1999), 27.
- [29] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *Phys. Rev.* **D60**, 124012 (1999);  
L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *Phys. Rev.* **D60**, 124013 (1999).
- [30] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *Annals of Phys.* **284**, 197 (2000), gr-qc/9906114.
- [31] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *J. Math. Phys.* **42**, 3 (2001), 1173.
- [32] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, in: *Differential Geometry and Its Applications*, Mathematical Publications, Vol. 3, Editors O. Kowalski, D.Krupka, J. Slovák, Proc. Conf. Opava (Czech Republic), Opava, August 2001, 401–413.
- [33] L.Fatibene, M.Francaviglia, *Natural and gauge natural formalism for classical field theories: a geometric perspective including spinors and gauge fields*, to be published by Kluwer Academic Publishers.
- [34] M. Ferraris, in: *Geometrical Methods in Physics*, Proceedings of the Conference on Differential Geometry and Its Applications, Nové Město na Moravě, Czechoslovakia, September 5-9, 1983.
- [35] M. Ferraris, M. Francaviglia, in: *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, Editor: M. Francaviglia, Elsevier Science Publishers B.V., (Amsterdam, 1991) 451.
- [36] M. Ferraris and M. Francaviglia, *Atti Sem. Mat. Univ. Modena*, **37** (1989), 61;  
M. Ferraris and M. Francaviglia, *Gen. Rel. Grav.*, **22**, (9) (1990), 965;  
M. Ferraris, M. Francaviglia, I. Sinicco, *Il Nuovo Cimento*, **107B**, N. 11, (1992), 1303;  
M. Ferraris, M. Francaviglia, *7th Italian Conference on General Relativity and Gravitational Physics*, Rapallo (Genoa), September 3–6, 1986.
- [37] M.Ferraris, J.Kijowski, *Gen. Relativ. Grav.* **14**, No.1 (1982), 37.
- [38] M. Ferraris, M. Francaviglia and O. Robutti, in: *Géométrie et Physique*, Proceedings of the *Journées Relativistes 1985* (Marseille, 1985), 112 – 125; Y. Choquet-Bruhat, B. Coll, R. Kerner, A. Lichnerowicz eds. (Hermann, Paris, 1987).
- [39] M.Ferraris, M.Francaviglia, V.Tapia, *J. Phys. A: Math. Gen.* **26** (1993), 433; V.Tapia, M.Ferraris, M.Francaviglia, *Il Nuovo Cimento* **103B**, N.4 (1989) 435.

- [40] M. Francaviglia, M. Raiteri, Class. Quantum Grav. **19**, (2002), 237 (gr-qc/0107074).
- [41] M. Godina, P. Matteucci, in press for J. Geom. Phys. (math.DG/0201235).
- [42] H.Goldschmidt, D.Spencer, J. Diff. Geom. **13** (1978) 455.
- [43] M.B. Greens, J.H. Schwarz, E.Witten, *Superstring Theory* (Cambridge University Press, England 1987);  
D. Lust, S. Theusen, *Lectures on String Theory* (Springer, Berlin, 1989);  
J.Polchinski, *String Theory* (Cambridge University Press, England 1998).
- [44] S. W. Hawking, C. J. Hunter, Class. Quantum Grav. **13**, (1996) 2735;  
S. W. Hawking, G. T. Horowitz, Class. Quantum Grav. **13**, (1996) 1487.
- [45] F.W.Hehl, P. von der Heyde, G.D.Kerlick, J.N.Nester, Reviews of Modern Physics, **48**, No. 3 (1976), 393.
- [46] M. Henneaux, B. Julia, S. Silva Nucl.Phys. **B563** (1999) 448 (hep-th/9904003);  
L.Fatibene, M. Francaviglia hep-th/0108112;  
G. Giachetta, L. Mangiarotti, G. Sardanashvily, math-ph/0305014;
- [47] D. Hilbert, *Königl. Gesel. Wiss. Göttingen. Nachr., Math. Phys. Kl.* (1915), 394;  
A. Einstein, *Sitzungsber. Preuss. Akad. Wiss. (Berlin)*, (1916), 1111,.
- [48] V. Iyer and R. Wald, Phys. Rev. D **50**, (1994), 846.
- [49] B. Julia, S. Silva, Class. Quantum Grav. **15**, (1998), 2173 (gr-qc/9804029).
- [50] B. Julia, S. Silva, Class. Quantum Grav. **17**, (2000), 4733 (gr-qc/0005127);  
S. Silva, Nucl. Phys. **B 558**, (1999), 391 (hep-th/9809109).
- [51] J. Katz, Class. Quantum Grav. **2**, (1985) 423;  
J. Katz, J. Bicak, D. Lynden-Bell, Phys. Rev. D **55** (10), (1997) 5957.
- [52] T.W.B.Kibble, J. Math. Phys. **2** (1961), 212;  
D.W. Sciama, *On the analogy between Charge and Spin in General Relativity*, in: Recent Development in General Relativity, Pergamon Press (London 1962), 415.
- [53] J. Kijowski, Gen. Relativ. Grav. **29** (1997), 307.
- [54] J. Kijowski, Gen. Rel. Grav. **9**(10) (1978), 857;  
M. Ferraris, J. Kijowski, *Rend. Sem. Mat. Univers. Politecn. Torino* **41**, (1983), 169;  
M. Ferraris, J. Kijowski, Gen. Rel. Grav. **14** (2) (1982), 165;  
M. Ferraris, M. Francaviglia, M. Raiteri, J. Math. Phys. **41**, 4, (2000) 1889.

- [55] I. Kolář, J. Geom. Phys. **1** (1984), 127.
- [56] I. Kolář, P.W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, (New York, 1993).
- [57] A. Komar, Phys. Rev. **113** (1959), 934.
- [58] Y.Kosmann, *Dérivées de Lie des spineurs*, Ann. Mat. Pura Appl. (4), **91**, (1972), 317.
- [59] P. Matteucci, gr-qc/0201079.
- [60] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [61] E. Nöther, Königl. Gesel. Wiss., Göttingen Nachr., Math. Phys. Kl. (1918), 235.
- [62] P.Olver, *Applications of Lie Groups to Differential Equations*, Springer Verlag, New York 1993.
- [63] T. Regge, C. Teitelboim, Annals of Physics (N.Y.) **88**, (1974), 286.
- [64] J.Rosen, Annals of Phys. **82** (1974), 54.
- [65] A. Trautman, in: *Gravitation: An Introduction to Current Research*, L. Witten ed. (Wiley, New York, 1962) 168;  
A. Trautman, Commun. Math. Phys., **6**, (1967), 248.
- [66] J. W. York, Foundation of Phys., **16** (3), (1986) 249.